A novel derivation for Kerr metric in Papapetrou gauge

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Abstract

We present a simple novel derivation, ab initio, of the equations appropriate for stationary axisymmetric spacetimes using the Papapetrou form of the metric (Papapetrou gauge). It is shown that using coordinates which preserve the Papapetrou gauge three separated solutions of the Ernst equations appear in the case of Kerr metric. In this context a parameter arises which represents topological defects induced by an infinite static string along the z axis. Finally, we discuss a simple solution that may be derived from the Kerr ansatz.

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1 Introduction

Contrary to the claim of Landau and Lifshitz that "there is no constructive analytic derivation of the Kerr metric that is adequate in its physical ideas and even a check of this solution of Einstein's equations involves cumbersome calcualtions" [1], Chandrasekhar [2] first derived and properly reduced the equations leading to the Kerr metric. He used the spheroidal coordinate η by setting $r = m + \sqrt{m^2 - a^2}\,\eta$ (here m is the mass and ma is the angular momentum), instead of the much more commonly used Boyer-Lindquist

(BL) coordinate r [3]. Although allowing a presentation of the metric in a simple (polynomial) form, this coordinate system leads to a confusing interpretation of the radial variable r (see [2], pg. 341,357). The Kerr metric is also commonly derived by using a method introduced by Ernst [4]. Namely, one shifts to prolate coordinates and again obtains a polynomial form characterized by the relation $m^2 - a^2 = 1$ (notice this implies $a^2 < m^2$) and recovers the standard expression performing a transformation to the BL coordinates. It is clear that this simple procedure does not allow the treatment of the case $a^2 > m^2$ but, perhaps less obviously, even the limit $q \to 1$ (here $q = a/m$), corresponding to the so called extreme ($a^2 = m^2$) Kerr solution, cannot be described in a physically satisfactory way. As a matter of fact in the literature (consult for example [5] and references therein) the following relation between cylindrical and spheroidal prolate coordinates is exploited: $\rho = m\sqrt{1-q^2}\sqrt{x^2-1}\sqrt{1-y^2},\ z = m\sqrt{1-q^2}\ xy$. This amounts to considering the unit lenght as depending on a, m. As a consequence, for the coordinate transformation quoted above to make sense when $q \to 1$, we must send $x \to \infty$. To put it in a somewhat fancy manner: in order to "see the Kerr metric" the observer must sit at the point at infinity. Moreover after this procedure we are left with no information about the requirements needed to obtain the extreme solution in the same spirit as the static one, namely by continuously modifying the parameters without changing the coordinates. This paper tries to overcome these problems.

The Papapetrou gauge [6] permits a description of the Kerr line element avoiding these shortcomings and we will show that in this case three distinct solutions appear that cannot be transfomed into each other by continuously varying the two parameters a and m. In section 2 we perform a novel derivation of the field equations while in section 3 we study the covariance of the Ernst equations for general transformation of coordinates. In particular we observe that in the Papapetrou gauge they are invariant in form under analytic coordinate transformations thereby proving that these latter (and only these) leave the line element invariant in form. In section 4 we write down the three different Kerr solutions and use this property to prove that no continuous variation of the parameters connect them (without changing the gauge). In Section 5 we study the Kerr solution in presence of a cosmic string. Section 6 collects some conclusions.

2 Derivation of the field equations for the Kerr metric

Our starting point is the axially symmetric line element in the form

$$ds^2 = e^\nu \left[(dx^1)^2 + (dx^2)^2 \right] + l d\varphi^2 + 2m d\varphi dt - f dt^2, \tag{1}$$

where $\nu = \nu(x^1, x^2)$, $l = l(x^1, x^2)$, $m = m(x^1, x^2)$, $f = f(x^1, x^2)$, φ is an angular coordinate , t the time coordinate, x^1, x^2 spatial coordinates and

$$fl + m^2 = \rho^2, \tag{2}$$

where ρ is the radius in a cylindrical coordinate system. The Ricci tensor $R_{\mu\nu}$ is defined in terms of Christoffel symbols by $R_{\mu\beta} = -\Gamma^\alpha_{\beta\mu,\alpha} + \Gamma^\alpha_{\beta\alpha,\mu} + \Gamma^\sigma_{\beta\alpha}\Gamma^\alpha_{\sigma\mu} - \Gamma^\sigma_{\beta\mu}\Gamma^\alpha_{\sigma\alpha}$. For the metric (1) the non identically vanishing Einstein's equations are: $R_{11} = R_{22} = R_{12} = R_{33} = R_{34} = R_{44} = 0$ (with coordinates $x^1, x^2, x^3 = \varphi, x^4 = t$). Thanks to (2) we can eliminate l and choose a coordinate system such that $\rho_{11} + \rho_{22} = \Delta\rho = 0$. We introduce a new function γ by $e^{2\gamma} = f e^\nu$ and the function ω by $\omega = \frac{m}{f}$. The Einstein's equations thus obtained are

$$\gamma_1 = -\frac{\Sigma\rho_1 + \Pi\rho_2}{4\rho({\rho_1}^2 + {\rho_2}^2)} + \frac{c}{2}\ ,\ \gamma_2 = \frac{\Sigma\rho_2 - \Pi\rho_1}{4\rho({\rho_1}^2 + {\rho_2}^2)} + \frac{d}{2}, \tag{3}$$

$$\nabla^2 f + \frac{f}{\rho^2}\left({\omega_\alpha}^2 f^2 - \frac{\rho^2}{f^2}{f_\alpha}^2 \right) = 0\ ,\ \tilde{\nabla}^2\omega + 2\omega_\alpha \frac{f_\alpha}{f} = 0. \tag{4}$$

A summation over α is implicit in equations (4) with $\alpha = 1, 2$, i.e. x^1, x^2, low indices denote partial derivatives and $\nabla^2 = \partial^2_{\alpha\alpha} + \frac{\rho_\alpha}{\rho}\partial_\alpha$, $\tilde{\nabla}^2 = \partial^2_{\alpha\alpha} - \frac{\rho_\alpha}{\rho}\partial_\alpha$, with

$$c = \frac{[2\rho_{12}\rho_2 + (\rho_{11} - \rho_{22})\rho_1]}{(\rho_1^2 + \rho_2^2)}\ ,\ d = \frac{[2\rho_{12}\rho_1 - (\rho_{11} - \rho_{22})\rho_2]}{(\rho_1^2 + \rho_2^2)}, \tag{5}$$

$$\Sigma = -\frac{\rho^2}{f^2}(f_1^2 - f_2^2) + f^2\left[\left\{ \left(\frac{m}{f}\right)_1 \right\}^2 - \left\{ \left(\frac{m}{f}\right)_2 \right\}^2 \right],$$

$$\Pi = -2\rho^2\frac{f_1 f_2}{f^2} + 2f^2\left(\frac{m}{f}\right)_1\left(\frac{m}{f}\right)_2.$$

In order to write the field equations in a complex form, some care must be taken. In fact, if we define [4] a potential $\tilde{\phi}$ such that

$$\omega_1 = -\frac{\rho}{f^2}\tilde{\phi}_2\ ,\ \omega_2 = \frac{\rho}{f^2}\tilde{\phi}_1, \tag{6}$$

the equation (4) involving $\tilde{\nabla}^2\omega$ becomes an identity and therefore cannot be taken as a field equation.
In any case, another equation can be drawn from the condition $\phi_{12} = \phi_{21}$ or $\omega_{12} = \omega_{21}$. The new equations for f and $\tilde{\phi}$ are

$$\nabla^2 f - \frac{1}{f}({f_\alpha}^2 - \tilde{\phi}_\alpha^2) = 0 \ , \ \nabla^2\tilde{\phi} - \frac{2}{f} f_\alpha \tilde{\phi}_\alpha = 0. \tag{7}$$

Once equations (7) are solved, the other functions γ and ω can be obtained by a simple integration.
Finally, we define a complex function ξ, with $f + \imath\tilde{\phi} = \frac{\xi-1}{\xi+1}$. It follows that equations (7) can be reduced to a single complex equation given by

$$(\xi\xi^* - 1)\left(\xi_{\alpha\alpha} + \frac{\rho_\alpha}{\rho}\xi_\alpha\right) = (\xi\xi^* - 1)\,\nabla^2\xi = 2\xi^*{\xi_\alpha}^2, \tag{8}$$

where "$*$" denotes complex conjugation. Summarizing, we have expressions (5) and equations (6), (8) and (3), with

$$\Sigma = -4\rho^2\frac{(\xi_1\xi_1^* - \xi_2\xi_2^*)}{(\xi\xi^* - 1)^2} \ , \ \Pi = -4\rho^2\frac{(\xi_1^*\xi_2 + \xi_1\xi_2^*)}{(\xi\xi^* - 1)^2}, \tag{9}$$

$$f = \frac{\xi\xi^* - 1}{(\xi + 1)(\xi^* + 1)} \ , \ \tilde{\phi} = -\imath\frac{(\xi - \xi^*)}{(\xi + 1)(\xi^* + 1)}. \tag{10}$$

3 Covariance of Ernst equations for a general coordinate transformation

In what follows we consider the behaviour of the Ernst equations (7) under a general coordinate transformation. Starting from the functions $f = f(x, y)$ and $\tilde{\phi} = \tilde{\phi}(x, y)$ we will choose a generic coordinate transformation $x = x(x', y')$ and $y = y(x', y')$. After simple but very tedious calculations, we find that in the new coordinates x', y' the equations (7) become

$$f\nabla^2 f(x', y') - f_A^2 + \tilde{\phi}_A^2 = 0, \ , \ f\nabla^2\tilde{\phi}(x', y') - 2\tilde{\phi}_A f_A = 0 \quad , \quad A = x', y' \tag{11}$$

if and only if the following conditions are imposed

$$A = B \ , \ C = 0 \ , \ \Delta x' = \Delta y' = 0, \tag{12}$$

$A = \left(\frac{\partial x'}{\partial x}\right)^2 + \left(\frac{\partial x'}{\partial y}\right)^2$, $B = \left(\frac{\partial y'}{\partial x}\right)^2 + \left(\frac{\partial y'}{\partial y}\right)^2$, $C = \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x} + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y}$.
It is easy to see that the solution of the system (12), thanks to $\frac{\partial^2 x'}{\partial x \partial y} =$

$\frac{\partial^2 x'}{\partial y \partial x}$, $\frac{\partial^2 y'}{\partial x \partial y} = \frac{\partial^2 y'}{\partial y \partial x}$, is given by the harmonic ansatz

$$\frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y} \ , \ \frac{\partial x'}{\partial y} = -\frac{\partial y'}{\partial x}. \tag{13}$$

The condition (13) can be simply integrated provided that $x + \imath y = F(x' + \imath y')$, where F is analytic. Therefore, to preserve the Papapetrou gauge, we must use analytic coordinate transformations. In the following section we will derive the Kerr solutions using the Papapetrou gauge.

4 Kerr solutions

4.1 Kerr solution with $a^2 < m^2$

Let us take spheroidal prolate coordinates defined in terms of the cylindrical ones by the analytic transformation

$$\rho = \sinh\mu \sin\theta \ , \ z = \cosh\mu \cos\theta. \tag{14}$$

In this adapted coordinate system we have $x^1 = \mu$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = t$ with the line element $ds^2 = f^{-1}\left[e^{2\gamma}\left(d\mu^2 + d\theta^2\right) + \rho^2 d\varphi^2\right] - f(dt - \omega d\varphi)^2$. It is easy to see that the complex function $\xi = p\cosh\mu + \imath q\cos\theta$ is a solution of (8) with p and q real constants satisfying $p^2 + q^2 = 1$. The metric functions f, γ and ω are given by

$$f = \frac{p^2 \cosh^2\mu + q^2 \cos^2\theta - 1}{(p\cosh\mu + 1)^2 + q^2\cos^2\theta} \ , \ \omega = 2\frac{q}{p}\frac{(p\cosh\mu + 1)\sin^2\theta}{\left[p^2\cosh^2\mu - 1 + q^2\cos^2\theta\right]},$$
$$e^{2\gamma} = \frac{\left(p^2\cosh^2\mu - 1 + q^2\cos^2\theta\right)}{k^2}. \tag{15}$$

The integration constant k is a new parameter that will be discussed later. At this stage we set $k^2 = p^2$ in equation (15) and define p and q as $p = 1/m$ and $q = a/m$, with $m^2 - a^2 = 1$. Performing an expansion of the metric at large distances one sees clearly that m can be identified with the mass and ma with the angular momentum of the source. Introducing the BL coordinates, defined in terms of the cylindrical ones by

$$\rho = \sqrt{r^2 + a^2 - 2mr}\ \sin\theta \ , \ z = (r - m)\cos\theta, \tag{16}$$

(the relation between r and μ is given by $r = \sqrt{m^2 - a^2}\ \cosh\mu + m$), the line element becomes

$$\begin{aligned} ds^2 &= (r^2 + a^2\cos^2\theta)\left(d\theta^2 + \frac{dr^2}{r^2 + a^2 - 2mr}\right) - dt^2 + \\ &+ (r^2 + a^2)\sin^2\theta d\varphi^2 + \frac{2mr}{r^2 + a^2\cos^2\theta}\left(dt + a\sin^2\theta d\varphi\right)^2. \end{aligned} \qquad (17)$$

The static limit is obtained from solution (15) by setting $a = 0$ and then $q = 0$ and $p = 1$ ($m = 1$). This is consistent with physical requirements because the parameters of the source are changed continuously. To obtain the extreme Kerr solution we must instead take the limit $a^2 = m^2$ and then $q = 1$, $p = 0$. But then, since $p = \frac{1}{m}$, $m^2 - a^2 = 1$ and therefore $m^2 \neq a^2$ provided that $m, a \neq \infty$. Besides, expression (15) for γ (with $k^2 = p^2$) diverges as $p \to 0$ and so does ω . We can also cast p and q in the form $p = \frac{\sqrt{m^2 - a^2}}{m}$, $q = \frac{a}{m}$. In this way, only a change in unit lenght occurs: distances are no longer measured in units $\sqrt{m^2 - a^2}$. We have again the relation (16) but in expression (14) ρ is substituted by $\rho/\sqrt{(m^2 - a^2)}$, z by $z/\sqrt{(m^2 - a^2)}$ and $k^2 = \frac{1}{m^2}$. Also, $\omega \to \sqrt{m^2 - a^2}\omega$. Owing to this fact, expression (14) is not defined as $a^2 \to m^2$ and in this limit the extreme Kerr solution does not emerge. On the contrary, thanks to (17), we can provide a unified description of the Kerr solutions. The coordinate transformation (16) is not analytic and the Papapetrou gauge breaks down. The condition $p^2 + q^2 = 1$ fixes the solution to be confined in the range $a^2 < m^2$. This fact is coordinate-independent, provided that the Papapetrou gauge is used. To show this, we use the relation (14) with the inverse given by $\cosh\mu = \frac{1}{2}\left[A + B\right]$, $\cos\theta = \frac{1}{2}\left[A - B\right]$ with $A = \sqrt{(z+1)^2 + \rho^2}$, $B = \sqrt{(z-1)^2 + \rho^2}$. The line element in cylindrical coordinates becomes

$$ds^2 = f^{-1}\left[\frac{e^{2\gamma}}{AB}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right] - f(dt - \omega d\varphi)^2, \qquad (18)$$

where f, ω and γ are given by equations (15) with the functions $\cosh\mu$ and $\cos\theta$ written in terms of ρ and z.

It follows that asking for the conservation of the Papapetrou gauge makes a unified description of the Kerr solution not achievable: three different ansätze are needed and it is impossible to join one to the other by a continuous variation of the parameters caracterizing the source. No such phenomenon occurs in the usual approach to the problem (see for example [5]) which relies on a relation between spheroidal prolate coordinates and cylindrical ones. More explicitly, first one sets $\rho = mp\sinh\mu\sin\theta, z =$

$mp\cosh\mu\cos\theta$ ($p = \frac{\sqrt{m^2-a^2}}{m}$) and then sends p to zero and μ to infinity. It is in this way that the extreme Kerr solution emerges. We now proceed to show how this result can be obtained in our chosen gauge.

4.2 Kerr solution with $a^2 = m^2$

To use the Papapetrou form of the metric we have to modify the usual spherical coordinates $x^1 = r$, $x^2 = \vartheta$, $x^3 = \varphi, x^4 = t$ with $\rho = r\sin\vartheta$, $z = r\cos\vartheta$. In fact, we will take $r = e^\nu$ with $-\infty < \nu < \infty$ so that $\rho = e^\nu \sin\vartheta$ and $z = e^\nu\cos\vartheta$. It is a simple matter to verify that the complex function $\xi = pe^\nu + \imath q\cos\vartheta$ satisfies (8) with the condition $q^2 = 1$. In the same way the complex function $\xi = p\cos\vartheta + \imath q e^\nu$ is a solution with $p^2 = 1$: it will be discussed in the Appendix. We emphasize that the condition $q^2 = 1$ is different from the appropriate one for the solution with $a^2 < m^2$ ($p^2+q^2 = 1$). Once the field equations are solved, the metric can again be expressed in spherical coordinates. The result is

$$
\begin{aligned}
ds^2 &= \frac{1}{k^2}\left[p^2 + \frac{1}{r^2} + \frac{2p}{r} + \frac{\cos^2\vartheta}{r^2}\right]dr^2 + \frac{1}{k^2}\left[p^2r^2 + 1 + 2pr + \cos^2\vartheta\right]d\vartheta^2 + \\
&+ \frac{\left[p^2r^2 + 2pr + 1 + \cos^2\vartheta\right]r^2\sin^2\vartheta}{\left(p^2r^2 - \sin^2\vartheta\right)}d\varphi^2 - \\
&- \frac{\left(p^2r^2 - \sin^2\vartheta\right)}{\left(p^2r^2 + 1 + 2pr + \cos^2\vartheta\right)}\left[dt - \frac{2q}{p}\frac{(pr+1)\sin^2\vartheta}{\left(p^2r^2 - \sin^2\vartheta\right)}d\varphi\right]^2 . \qquad (19)
\end{aligned}
$$

Note that in these coordinates the point $r = 0$ represents the location of two coinciding horizons, i.e. the origin of the spatial coordinates.

4.3 Kerr solution with $a^2 > m^2$

We consider spheroidal oblate coordinates $\rho = \cosh\mu\cos\theta$, $z = \sinh\mu\sin\theta$, with inverse given by $\cosh\mu = \frac{1}{2}[A + B]$, $\cos\theta = \frac{1}{2}[A - B]$ where $A = \sqrt{z^2 + (\rho+1)^2}$, $B = \sqrt{z^2 + (\rho-1)^2}$. Again, it is not difficult to see that the complex function $\xi = p\sinh\mu + \imath q\sin\theta$ is a solution of (8) with the condition $q^2 - p^2 = 1$. Setting $p = \frac{1}{m}$, $q = \frac{a}{m}$, this means $a^2 - m^2 = 1$. Equivalently, we can choose $p = \frac{\sqrt{a^2-m^2}}{m}$, $q = \frac{a}{m}$. From the relation $q^2 - p^2 = 1$ it follows that p becomes pure imaginary if $a \to 0$ and we cannot take the static limit accordingly with the considerations above. Concerning

the line element $ds^2 = \frac{e^{2\gamma}}{f}\left[d\mu^2 + d\theta^2\right] + \frac{\rho^2}{f}d\varphi^2 - f(dt - \omega d\varphi)^2$, we get

$$f = \frac{p^2\sinh^2\mu + q^2\sin^2\theta - 1}{(p\sinh + 1)^2 + q^2\sin^2\theta}\ ,\ \omega = \frac{2q}{p}\frac{(p\sinh\mu + 1)\cos^2\theta}{[p^2\sinh^2\mu - 1 + q^2\sin^2\theta]}\ ,$$
$$e^{2\gamma} = \frac{p^2\sinh^2\mu - 1 + q^2\sin^2\theta}{p^2}. \tag{20}$$

To cast the metric in the form (17) we use the relation $r = \sqrt{a^2 - m^2}\sinh\mu + m$ and perform the rotation $\theta \to \theta - \pi/2$. We then see that it is impossible to go from the "oblate" solution to the "prolate" one in a mathematically and physically reasonable manner.

5 Kerr solutions with topological defect

As a final illustration of the significance of the Papapetrou gauge we show how it can be used to study in simple way Kerr solutions in the presence of a cosmic string. All the three metrics we obtained (without setting $k^2 = p^2$) have, after rescaling by a factor p^2/k^2, a common asymptotic behaviour. Explicitly (setting G=c=1)

$$ds^2 = d\rho^2 + dz^2 + C^2\rho^2 d\varphi^2 - dt^2, \tag{21}$$

where $k^2/p^2 = C^2$. It is a well known fact [7] that, when $C < 1$, the space-time (21) is a solution of Einstein's equations with stress-energy tensor

$$T_{\mu\nu} = \mu\delta(x)\delta(y)diag(1, 0, 0, -1)\ ,\ C = 1 - 4\mu. \tag{22}$$

Expression (22) represents a string on the z axis with constant mass density μ. The parameter C represents a topological defect with angle deficit $2\pi - 2\pi C = 8\pi\mu$. It is also known [8] that in the limit $\rho \to 0$ the quantity

$$\Delta\Phi(\rho) = 2\pi - \frac{\int_0^{2\pi}\sqrt{g_{\varphi\varphi}}d\varphi}{\int_0^{\rho}\sqrt{g_{\rho\rho}}d\rho}, \tag{23}$$

is directly related to the energy density per unit lengh of the string. If $\Delta\Phi(0) = 0$ the topological defect disappears but for the three metrics that we are considering we have $\Delta\Phi(0) = 2\pi(1 - C)$. Consider first the "prolate" case (18): performing a Taylor expansion in the neighbourhood of the z axis ($\rho = 0$), we obtain (outside the horizon) $\sqrt{g_{\varphi\varphi}} = \frac{\rho\sqrt{p^2z^2+1+2pz+q^2}}{p\sqrt{z^2-1}} + o(\rho^2)$ and $\sqrt{g_{\rho\rho}} = \frac{\sqrt{p^2z^2+1+2pz+q^2}}{Cp\sqrt{z^2-1}} + o(\rho)$. It follows that $\Delta\Phi(0) = 2\pi(1 - C) = 8\pi\mu$.

In a similar way the same result is obtained for the metrics (19) and (20). Finally, when the parameter C is used, the metric (17) in BL coordinates becomes

$$
\begin{aligned}
& ds^2 = \frac{\Sigma}{C^2}\left(d\theta^2 + \frac{dr^2}{\Delta}\right) + (r^2 + a^2)\sin^2\theta d\varphi^2 - dt^2 + \\
& + \frac{2mr}{\Sigma}\left(dt + a\sin^2\theta d\varphi\right)^2, \\
& \Sigma = r^2 + a^2\cos^2\theta \quad , \quad \Delta = r^2 + a^2 - 2mr,
\end{aligned} \tag{24}
$$

which describes the Kerr solutions with a static string on the z axis.

6 Conclusions

In this paper we have presented a simple novel derivation of equations appropriate for a stationary axisymmetric space-time using Papapetrou gauge. The Kerr solutions in this gauge are disconnected: it is impossible to go from one to the other by continuously changing the parameters. A quantity is introduced which represents a topological defect induced from a static infinitely long cosmic string on the z axis.

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APPENDIX

Using spherical analytic coordinates ($r = e^\nu$) we have the solution $\xi = p\cos\vartheta + \imath q e^\nu$ with $p^2 = 1$. For simplicity we take $p = 1$. The metric is

$$
\begin{aligned}
& ds^2 = \frac{\left[(1+\cos\vartheta)^2 + q^2r^2\right]}{k^2r^2}dr^2 + \frac{\left[(1+\cos\vartheta)^2 + q^2r^2\right]}{k^2}d\vartheta^2 + \\
& + r^2\sin^2\vartheta\frac{\left[(1+\cos\vartheta)^2 + q^2r^2\right]}{q^2r^2 - \sin^2\vartheta}d\varphi^2 - \\
& - \frac{q^2r^2 - \sin^2\vartheta}{(1+\cos\vartheta)^2 + q^2r^2}\left[dt - \frac{2qr^2(1+\cos\vartheta)}{q^2r^2 - \sin^2\vartheta}d\varphi\right]^2.
\end{aligned} \tag{25}
$$

Putting $k = q$, we have in the asymptotic limit

$$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left[dt - \frac{2}{q}(1 + \cos\vartheta)d\varphi\right]^2. \qquad (26)$$

This expression has the same asymptotic behaviour of the N.U.T solution [9]. Recently, the general Kerr-N.U.T. solution has been obtained (see [10]) in a very simple form and its uniqueness has been established for spacetimes admitting separable equations of motion.

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